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Roll No. | | | | | | | | | | | |

2019

S. No. of Question Paper : **2248**

Unique Paper Code : **32351201**

Name of the Paper : **Real Analysis**

Name of the Course : **B.Sc. (H) Mathematics**

Semester : **II**

Duration : **3 Hours** Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

All questions are compulsory.

There are internal choices in Q. Nos. **2-5**.

1. Prove or disprove : 6×2½=15

(*a*) If $x \in R$, $x > 0$, then $\frac{1}{x} > 0$.

(*b*) If s is an upper bound of a non-empty set S such that $s \in S$, then $s = \sup S$.

(*c*) A sequence (x_n) satisfying $\lim (|x_{n+1} - x_n|) = 0$ is convergent.

P.T.O.

(*d*) $\lim ((a^n + b^n)^{1/n}) = b$, where $0 < a < b$.

(*e*) The series $\sum_{n=1}^{\infty} (\cos nx)$ converges for all $x \in \mathbf{R}$.

(*f*) $\sum_{n=1}^{\infty} \frac{n2^n}{(n^2+1)}$ is a convergent series.

2. Answer any *three* parts : 3×5=15

(*a*) State and prove the Density Theorem for real numbers.

(*b*) (*i*) Let $a, b \in \mathbf{R}$ and suppose that for every $\varepsilon > 0$, we have $a \leq b + \varepsilon$. Show that $a \leq b$.

(*ii*) Let S be a non-empty subset of **R**. Show that $u \in \mathbf{R}$ is an upper bound of S if and only if the conditions $t \in \mathbf{R}$, $t > u$ imply $t \notin S$.

(*c*) Let S be a non-empty bounded set in **R** and let $b < 0$. Prove that $\inf(bS) = b(\sup S)$ and $\sup(bS) = b(\inf S)$.

(*d*) If S is a non-empty subset of **R**, show that S is bounded if and only if there exists a closed and bounded interval I of **R** such that $S \subseteq I$.

3. Answer any *three* parts : 3×5=15

(*a*) Find the limit of the following sequences whose *n*th term is given by :

(*i*) $x_n = \frac{n}{b^n}$, where $b > 1$

(*ii*) $y_n = \frac{\sin n}{n} + \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)$.

(*b*) Prove that if a sequence (x_n) is increasing and bounded above, then it converges to u where u is the least upper bound of the set $\{x_n : n \in \mathbf{N}\}$.

(*c*) If a sequence (x_n) of real numbers converges to a real number x, prove that every subsequence (x_{n_k}) of (x_n) converges to x.

(*d*) State the Cauchy Convergence Criterion for sequences. Use it to show that the sequence (x_n) defined by

$$x_n = \frac{1}{1!} - \frac{1}{2!} + \ldots\ldots + \frac{(-1)^{n+1}}{n!}$$

is convergent.

4. Answer any *three* parts : 3×5=15

(*a*) Use the integral test to check the convergence of the series :

$$\sum_{n=2}^{\infty} \frac{1}{n \log n}.$$

(*b*) When do we say that the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent ? Show that the series :

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(\sqrt{n+1} - \sqrt{n}\right)$$

is absolutely convergent.

(*c*) Test the convergence of the following series :

(*i*) $\sum_{n=1}^{\infty} \frac{n^{n^2}}{(n+1)^{n^2}}$

(*ii*) $\sum_{n=1}^{\infty} (-1)^n \frac{(n!)^2}{(2n)!}$.

(*d*) (*i*) Find all $x \in R$ for which the series $\sum_{n=1}^{\infty} e^{nx}$ converges.

(*ii*) Show that the series $\sum_{n=1}^{\infty} \log\left(\frac{n}{n+1}\right)$ is divergent.

5. (*a*) (*i*) Let X and Y be non-empty sets and let $h : X \times Y \to \mathbf{R}$ have bounded range in **R**. Let $f : X \to \mathbf{R}$ and $g : Y \to \mathbf{R}$ be defined by

$$f(x) = \sup \{h(x, y) : y \in Y\},$$

$$g(y) = \inf \{h(x, y) : x \in X\}.$$

Prove that :

$$\sup \{g(y) : y \in Y\} \leq \inf \{f(x) : x \in X\}.$$

(*ii*) Give an example of a set which has exactly two limit points. 4,1

Or

(*i*) Show that for any real numbers p, q and rational number r such that $r < p + q$, there exist rational numbers $r_1 < p$ and $r_2 < q$ such that $r = r_1 + r_2$.

(*ii*) Provide a bijection between **N** and the set of all odd integers greater than 49. 3,2

P.T.O.

(b) (i) If $\lim (x_n) = x (\neq 0)$, prove that there is a positive number A and a natural number N such that $|x_n| > A$ for all $n \geq N$.

(ii) Is the sequence (x_n) where

$$x_n = \frac{n^3 + 3n^2}{n + 1} - n^2$$

bounded ? Justify. 3,2

Or

Is the sequence (x_n) where

$$x_n = \frac{n^2}{n^3 + n + 1} + \frac{n^2}{n^3 + n + 2} + \ldots\ldots + \frac{n^2}{n^3 + 2n}$$

convergent ? If yes, find its limit. 5

(c) State the Alternating Series Test. Show that the series :

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 1}$$

is conditionally convergent. 5

Or

If the series $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ are convergent, then prove that the series

$$\sum_{n=1}^{\infty} a_n b_n$$

is convergent where $a_n \geq 0$ and $b_n \geq 0$ for all $n \in \mathbf{N}$.

Hence or otherwise show that the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is convergent whenever $\sum_{n=1}^{\infty} a_n^2$ is convergent. 5